School of Mathematics Spring 2019

QUANTUM COMPUTATION

Exercise sheet 1

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1. **Revision.**

   (a) Imagine we have a quantum state $|\psi\rangle$ of n qubits, where $|\psi\rangle = \sum_{x \in \{0,1\}^n} \alpha_x |x\rangle$, and we measure the first qubit of $|\psi\rangle$ in the computational basis. What is the probability that the measurement outcome is 1, in terms of the α_x coefficients?
   Answer: $\sum_{x, x_1 = 1} |\alpha_x|^2$.

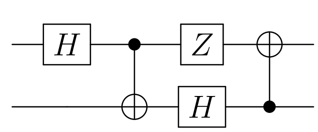
   (b) What is the state of the system after the measurement?
   Answer: $\frac{1}{\sqrt{\sum_{x, x_1=1} |\alpha_x|^2}} \sum_{x, x_1 = 1} \alpha_x |x\rangle$.

   (c) Let M be the matrix defined by $M = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ -1 & 1 \end{pmatrix}$. Is M unitary?
   Answer: No, because $M^\dagger M \neq I$.

   (d) Write down the matrix corresponding to the operator $H \otimes H$, in the computational basis, where H is the Hadamard operator.
   Answer:
   $$\frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}.$$

2. **The quantum circuit model.**

   (a) Consider the following quantum circuit C:

   i. Calculate the matrix of the unitary operation U corresponding to C, with respect to the computational basis.

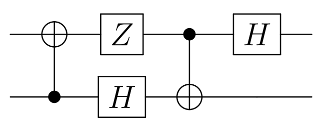
Answer: The answer can be obtained either by just multiplying out the matrices corresponding to the gates, or by tracking each computational basis state through the circuit, e.g.:

$$\begin{aligned}|0\rangle|0\rangle &\mapsto \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)|0\rangle \mapsto \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)\\ &\mapsto \frac{1}{2}(|0\rangle(|0\rangle + |1\rangle) - |1\rangle(|0\rangle - |1\rangle))\\ &\mapsto \frac{1}{2}(|00\rangle + |01\rangle - |10\rangle + |11\rangle).\end{aligned}$$

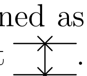
The final answer is

$$U = \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & -1 & 1 \\ -1 & -1 & 1 & 1 \\ 1 & -1 & 1 & -1 \end{pmatrix}.$$

ii. Write down a quantum circuit corresponding to the inverse operation U^{-1}.
Answer: As each gate in the circuit is its own inverse, U^{-1} can be implemented by running the circuit in reverse order, i.e.:

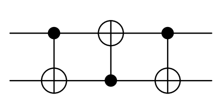


iii. If C is applied to the initial state $|0\rangle|0\rangle$ and is followed by a measurement of each qubit in the computational basis, what is the distribution on measurement outcomes?
Answer: The distribution on measurement outcomes is obtained by squaring the first column of U, and is hence uniform on $\{0,1\}^2$.

(b) The SWAP gate for 2 qubits is defined as SWAP$|x\rangle|y\rangle = |y\rangle|x\rangle$ for $x, y \in \{0,1\}$ and is denoted by the circuit element . Show that SWAP can be implemented as a product of CNOT gates and write down the corresponding circuit.
Answer: The matrix for SWAP in the computational basis is

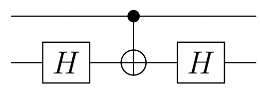
$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

By direct calculation, the following circuit corresponds to the same matrix:



(c) Show that a CZ gate can be implemented using a CNOT gate and Hadamard gates and write down the corresponding circuit.

Answer: Recall from Quantum Information Theory that $Z = HXH$. As CNOT is a controlled-X operation, we would expect that $CZ = (I \otimes H)\text{CNOT}(I \otimes H)$. And indeed this is the case, as can be verified from writing out the matrices and multiplying them together. The corresponding circuit is



(d) The classical OR gate takes as input a pair of bits $x, y \in \{0, 1\}$ and outputs 1 if either x or y is equal to 1, and 0 otherwise. Use the generic construction of reversible functions discussed in the lecture notes to write down a unitary operation on 3 qubits which corresponds to a reversible implementation of the OR gate.

Answer: Following the same construction as for AND, we obtain the map $|x\rangle|y\rangle|z\rangle \mapsto |x\rangle|y\rangle|z \oplus (x \text{ OR } y)\rangle$. Written explicitly as a matrix with respect to the computational basis, this is

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

3. **The Bernstein-Vazirani algorithm.**

A parity function $f_s : \{0, 1\}^n \to \{0, 1\}$, for some $s \in \{0, 1\}^n$, is a function of the form $f_s(x) = x \cdot s$, where the inner product is taken modulo 2. For example, with $n = 3$, $f_{110}(x)$ is the function $x_1 \oplus x_2$.

(a) Show that f_s is a balanced function for all $s \neq 0^n$.

Answer: We have $f_s(x) = \sum_i x_i s_i \bmod 2$. If $s \neq 0^n$, then there exists i such that $s_i \neq 0$. So, for all x, $f_s(x) \neq f_s(x^i)$, where x^i is the string obtained from x

by inverting bit i. Hence f_s is balanced.

(b) Imagine we apply the circuit for the Deutsch-Jozsa algorithm with the oracle U_{f_s}. Show that the measured output is precisely the string s.

Answer: The final state in the Deutsch-Jozsa algorithm is

$$\sum_{y \in \{0,1\}^n} \frac{1}{2^n} \left(\sum_{x \in \{0,1\}^n} (-1)^{f_s(x) + x \cdot y} \right) |y\rangle.$$

We have

$$\sum_{x \in \{0,1\}^n} (-1)^{f_s(x) + x \cdot y} = \sum_{x \in \{0,1\}^n} (-1)^{x \cdot s + x \cdot y} = \sum_{x \in \{0,1\}^n} (-1)^{x \cdot (s+y)}.$$

By the same argument as part (a), this evaluates to zero unless $s + y = 0^n \pmod 2$, or in other words unless $s = y$.

(c) Consider the following problem: given oracle access to a parity function f_s, determine s using the minimal number of queries to f_s.

i. Conclude from (b) that there is a quantum algorithm that solves this problem with one query to f_s.

Answer: We perform the Deutsch-Jozsa algorithm, using the oracle U_{f_s}, and measure the final result. The answer is s with certainty and the algorithm uses one query to U_{f_s} and hence one query to f_s.

ii. Give an exact bound on the number of queries to f_s required for a classical algorithm to solve the problem with certainty.

Answer: Each classical query has two outcomes, so reduces the space of possibilities for s by at most a factor of $1/2$. As there are 2^n possible strings s, the classical algorithm must make at least n queries. This is tight, because we can evaluate f_s on the strings $x^{(i)}$, $i = 1, \dots, n$ where $x^{(i)}$ is 1 at position i, and 0 elsewhere. Then $f_s(x^{(i)}) = s_i$, so each query reveals one bit of s.

4. **Simulation of various kinds. (Optional)**

(a) Show that the phase oracle U_f as defined in the lecture notes cannot be used to implement the bit oracle O_f in general, even if f only has 1 bit output.

Answer: Consider the two functions on one bit $f(x) = 0$ and $f(x) = 1$. Then in the first case, $U_f|x\rangle = |x\rangle$, and in the second case $U_f|x\rangle = -|x\rangle$; thus either $U_f = I$ or $U_f = -I$. These two operations are indistinguishable by any operations we might perform around them, because they only differ by a global phase of -1.

But in the case of the bit oracle $O_f|x\rangle|y\rangle = |x\rangle|y \oplus f(x)\rangle$, these two functions are indeed distinguishable (we could simply query O_f on $x = 0$). So U_f cannot be used to implement O_f in general.

(b) Imagine we are given a quantum circuit on n qubits which consists of poly(n) gates picked from the (universal) set $\{H, X, \text{CNOT}, T\}$, followed by a final measurement of all the qubits. Assume that at each step in the computation the quantum state is unentangled (i.e. is a product state of the n qubits). Show that the circuit can be simulated efficiently classically: that is, there is an efficient classical algorithm for exactly sampling from the probability distribution on the final measurement outcomes.

Answer: Imagine we start with a product state $|\psi_1\rangle|\psi_2\rangle \dots |\psi_n\rangle$. A description of this state can be written down in $O(n)$ space by writing down a description of each state $|\psi_i\rangle$ separately. We simulate the effect of each gate in the circuit on this state in turn. If we have H, X or T on qubit i, this can be done by multiplying $|\psi_i\rangle$ by the corresponding matrix, and updating the description of $|\psi_i\rangle$ accordingly. On the other hand, the CNOT gate involves two qubits i, j. So, once the gate has been applied, we need to find a new product state representation for the state of these qubits. This can be achieved by solving a system of equations in 4 variables corresponding to the amplitudes of the product states. At the end of the circuit, we have some product state of n qubits. To simulate sampling from the distribution on final outcomes x, we can sample each bit x_i from the distribution corresponding to state $|\psi_i\rangle$.