School of Mathematics **Spring 2018**

QUANTUM COMPUTATION

Exercise sheet 6

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1. **Shor's 9 qubit code.** Imagine we encode the state $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$ using Shor's 9 qubit code, and then an X error occurs on the 8th qubit of the encoded state $|E(\psi)\rangle$.

   (a) Write down the state following the error.
   Answer:

$$\frac{1}{2\sqrt{2}}(\alpha(|000\rangle+|111\rangle)(|000\rangle+|111\rangle)(|010\rangle+|101\rangle)+\beta(|000\rangle-|111\rangle)(|000\rangle-|111\rangle)(|010\rangle-|101\rangle)).$$

   (b) We now decode the encoded state, starting by applying the bit-flip code decoding algorithm. What are the syndromes returned by the measurements in the algorithm?
   Answer: Using the table in the lecture notes, the syndromes are 00, 00, 10.

   (c) Now imagine that $|E(\psi)\rangle$ is affected by two X errors, on the 7th and 8th qubits. What are the syndromes returned this time? What state does the decoding algorithm output?
   Answer: Now the syndromes are 00, 00, 01. The decoding algorithm thus thinks there has been an X error on the 9th qubit. So it "corrects" this by applying an X operation on this qubit, to give the state

$$\frac{1}{2\sqrt{2}}(\alpha(|000\rangle+|111\rangle)(|000\rangle+|111\rangle)(|000\rangle+|111\rangle)-\beta(|000\rangle-|111\rangle)(|000\rangle-|111\rangle)(|000\rangle-|111\rangle)).$$

   Note that β now has a minus sign in front of it. After the bit-flip decoding, we are left with $\alpha|+++\rangle - \beta|---\rangle$, which is then decoded to $\alpha|0\rangle - \beta|1\rangle$.

   (d) Which patterns of X errors are corrected by Shor's 9 qubit code?
   Answer: If there is at most one X error in each block of 3 qubits, these will be corrected properly. We have just seen that, if two errors occur in one block, the sign of β will be flipped, but the state is not otherwise affected; a similar argument holds for 3 errors in one block. So the output state will be correct if the number of blocks in which at least two errors occur is even (as then β will eventually be left unchanged).