NP-completeness

(or how to prove that problems are probably hard)

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- Why? Proving that a task is impossible can be helpful information, as it stops us from trying to complete it.
- During this lecture we'll take an informal approach to discussing this, and computational complexity in general – see the references at the end for more detail.

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- As N is specified by $O(\log N)$ bits, this algorithm runs in time exponential in the input size.

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- PRIMALITY: decide whether an integer is prime;
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The set of decision problems which have algorithms with runtime polynomial in the input size is known as P.

- So we think of P as the class of decision problems which can be solved efficiently.

Formalities

Some notes about formalising this notion (which we'll largely ignore for the rest of this lecture):

- A decision problem can be formally identified with a language, i.e. a subset $\mathcal{L} \subseteq \{0,1\}^*$, where $\{0,1\}^*$ is the set of bit-strings of arbitrary length.
- Each input bit-string x such that $x \in \mathcal{L}$ corresponds to an input such that the answer should be "yes"; all strings $x \notin \mathcal{L}$ correspond to inputs such that the answer should be "no".

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- The notion of "algorithm" should also be defined formally, in terms of Turing machines. However, we omit the details for this lecture.

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- We say that $\mathcal{L}_1$ reduces to $\mathcal{L}_2$ if such a transformation exists.
- If $\mathcal{L}_2 \in \mathrm{P}$, and $\mathcal{L}_1$ reduces to $\mathcal{L}_2$, then $\mathcal{L}_1 \in \mathrm{P}$.

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- Here the answer is indeed yes.

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- If the answer is "yes", we should accept a correct proof; but if the answer is "no", we should not be fooled by any incorrect proof.
- It is clear that $\mathrm{P} \subseteq \mathrm{NP}$, as if we can solve a problem in polynomial time, we can efficiently verify a claimed solution we are given: we just ignore it, and solve the problem ourselves.
- But whether or not $\mathrm{P} = \mathrm{NP}$ (aka the P vs. NP question) is the biggest unsolved problem in computer science!

More on NP

- The initials “NP” stand for Nondeterministic Polynomial (for reasons beyond the scope of this lecture. . .), and not Non-Polynomial.
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- Indeed, the P vs. NP question precisely asks whether all problems in NP have polynomial-time algorithms.
- Resolving P vs. NP would win you everlasting fame (as well as $1M from the Clay Mathematics Institute).
- Although we don't know whether P $=$ NP, most people consider this very unlikely, as it would imply that whenever we have an efficient algorithm to verify a “yes” solution to a decision problem, we also have an efficient algorithm to solve the problem.

NP-hardness and NP-completeness

- We say that a decision problem $\mathcal{L}$ is NP-hard if, for every problem $\mathcal{L}' \in$ NP, there is a polynomial-time reduction from $\mathcal{L}'$ to $\mathcal{L}$.
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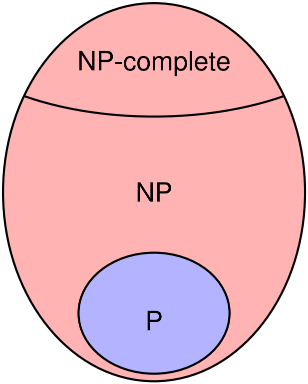
- We say that a problem $\mathcal{L}$ is NP-complete if $\mathcal{L}$ is NP-hard and $\mathcal{L} \in$ NP. Informally, NP-complete problems are the hardest problems in NP.

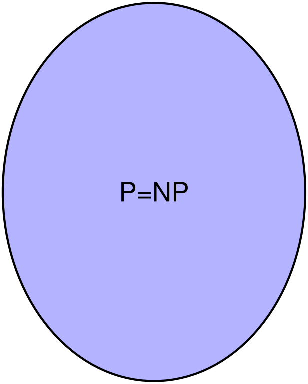
- It is not obvious that any NP-complete problems should exist...

P and NP in pictures

The picture if P≠NP:



The picture if P=NP:



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The CIRCUIT SAT (short for "satisfiability") problem is defined as follows.

- The input to the problem is a circuit (i.e. a sequence of AND, OR and NOT gates connected by wires in some order).
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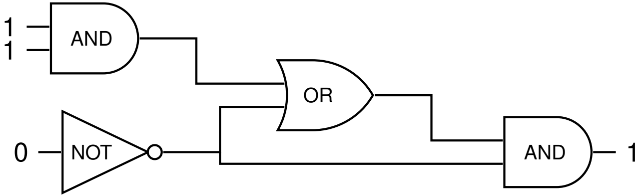

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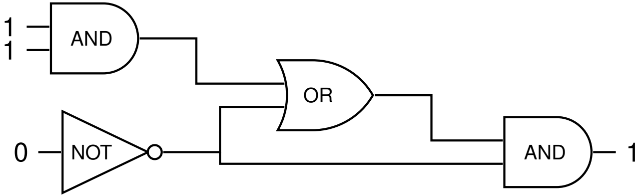

CIRCUIT SAT is in NP: if the answer is "yes", and we are given a claimed input such that the output is 1, we can simulate the circuit to check it.

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- If there exists a proof that the answer should be "yes", this corresponds to an input to the circuit such that the output is 1; otherwise, there is no such input.
- So, if we can solve CIRCUIT SAT, we can decide which of these is the case. □

More NP-complete problems

- Now that we know that Circuit SAT is NP-complete, we can use this to prove that other problems are also NP-complete.
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- What does this mean?

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For example:

$$(x_2 \vee x_1 \vee \neg x_3) \wedge (x_3 \vee \neg x_1) \wedge (\neg x_2 \vee x_3 \vee x_4)$$

is an instance of 3-SAT. It is satisfied by e.g. $x_1 = 0, x_2 = 1, x_3 = 0, x_4 = 1$.

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- It turns out that 3-SAT is actually NP-complete.

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- For each gate, there exists an assignment to the variables satisfying the clauses if and only if the gate behaves correctly.
- Finally, we have a clause containing a single variable, which is satisfied if and only if the output wire of the circuit is set to 1.

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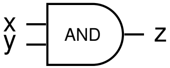
- The construction performs the mapping:



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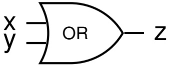

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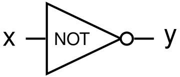

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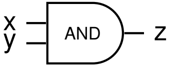

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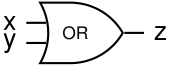

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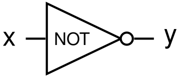

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- For example, $y = \neg x$ if and only if $(x \vee y) = 1$ and $(\neg x \vee \neg y) = 1$.

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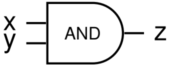

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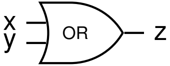

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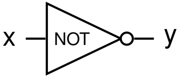

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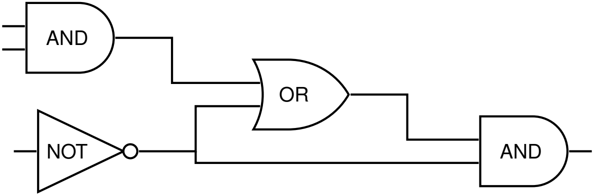
- Claim: All the clauses are satisfied if and only if all the gates work properly, and the output of the circuit is 1.

Example

Imagine we want to solve CIRCUIT SAT for the following circuit:



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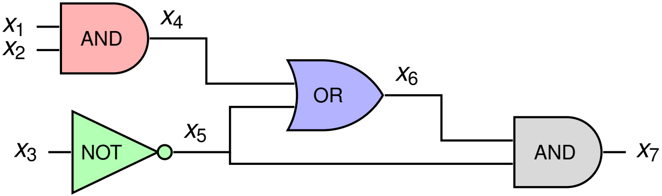

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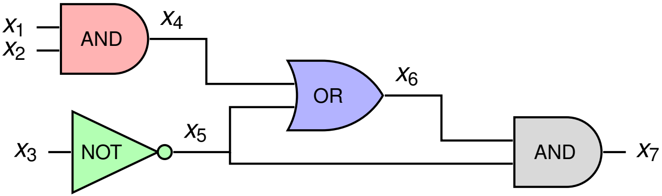

This maps to the following formula:

$$
\begin{aligned}
& (x_4 \vee \neg x_1 \vee \neg x_2) \wedge (\neg x_4 \vee x_1) \wedge (\neg x_4 \vee x_2) \\
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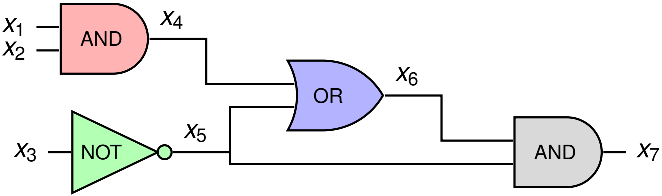

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The formula is satisfiable, so the original circuit is too.

Another NP-complete problem: 3-COLOURING

- We will now show NP-completeness of another problem, which is apparently quite different: graph colouring.

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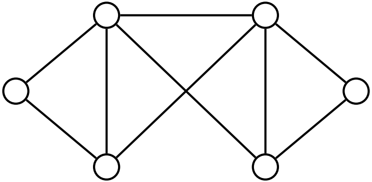

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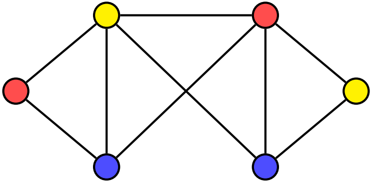

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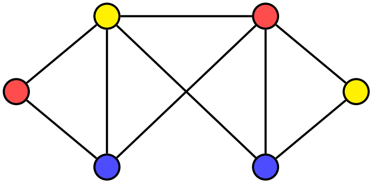

3-COLOURING is in NP because, if someone gives us a claimed colouring of a graph, we can check it efficiently.

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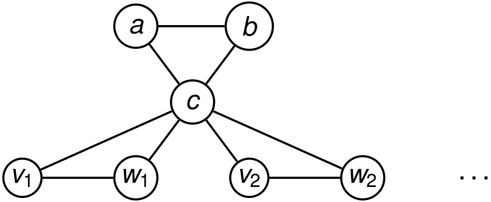
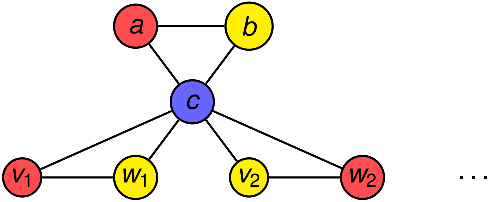
- We start by having a pair of vertices v_i, w_i for each variable x_i in the formula. Each of these vertices is connected to a central vertex c, which is connected in turn to two other vertices a and b.



Proof that 3-COLOURING is NP-complete (sketch)

- Imagine (without loss of generality) that vertices a, b and c are coloured red, yellow and blue.
- Then all of the pairs of vertices v_i, w_i must be coloured red and yellow (one of them red, and the other yellow).



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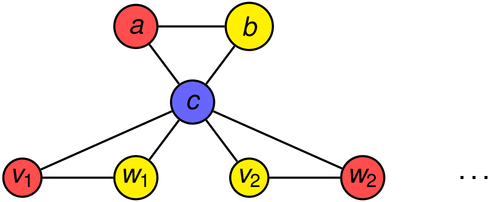

- This will be used to encode whether the i'th variable x_i is 0 or 1 in some assignment to the original formula.
- If v_i is red and w_i is yellow, this will correspond to $x_i = 0$; if v_i is yellow and w_i is red, this will correspond to $x_i = 1$.

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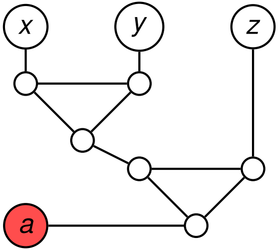

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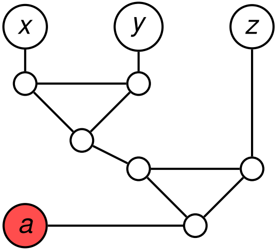

- Claim: There is a valid 3-colouring of the internal (unlabelled) vertices if and only if at least one of x, y, z is not coloured red.

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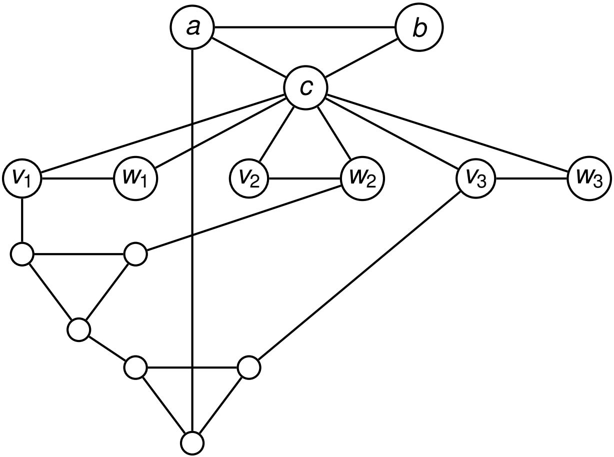
- Claim: Any valid colouring of the graph corresponds to an assignment to the variables such that all clauses are satisfied.
- This means that determining whether the graph is 3-colourable allows us to determine whether the formula is satisfiable, so 3-COLOURING is NP-complete.

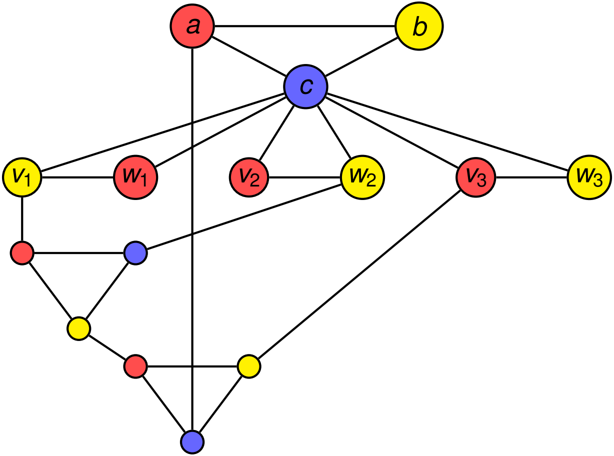
Example

The graph corresponding to the formula $(x_1 \vee \neg x_2 \vee x_3)$ is:



Example

The graph can be coloured properly, corresponding to the original formula having a satisfying assignment. One such colouring:



The colouring shown corresponds to assigning $x_1 = 1$, $x_2 = 0$, $x_3 = 0$.

Other NP-complete problems

A vast number of other problems have also been proven to be NP-complete, many of which are very important in science, engineering and business.

For example:

- Timetable scheduling
- Packing and covering problems
- Finding longest paths
- Solving systems of quadratic equations
- Partitioning problems
- Finding the longest common subsequence of two strings
- Many games and puzzles, e.g. generalised Sudoku and Lemmings
- Integer programming (see later in this course)
- …

Summary

- The theory of NP-completeness allows us to make rigorous the intuition that some problems are intrinsically hard.
- If a problem is NP-complete, this is good evidence that there is no efficient (polynomial-time) algorithm to solve it in the worst case.
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2. Find an efficient algorithm which outputs an approximate solution (see COMS31900: Advanced Algorithms for more);
3. Prove P=NP and win a million dollars.

Further Reading

- Introduction to Algorithms
  T. H. Cormen, C. E. Leiserson, R. L. Rivest and C. Stein.
  MIT Press/McGraw-Hill, ISBN: 0-262-03293-7.
  - Chapter 34 – NP-completeness
- Algorithms
  S. Dasgupta, C. H. Papadimitriou and U. V. Vazirani
  `http://www.cse.ucsd.edu/users/dasgupta/mcgrawhill/`
  - Chapter 8 – NP-complete problems
- Algorithms lecture notes, University of Illinois
  Jeff Erickson
  `http://www.cs.uiuc.edu/~jeffe/teaching/algorithms/`
  - Lecture 29 – NP-Hard Problems

Biographical notes

Stephen Cook (b. 1939)

- An American-Canadian mathematician who invented the notion of NP-completeness in a seminal paper in 1971.
- After this, many important problems were swiftly proven to be NP-complete.
- Cook won the Turing Award in 1982.
- Also has a computational complexity class named after him (SC).

Pic: Wikipedia

Biographical notes

Leonid Levin (b. 1948)

- Levin is a Soviet-American computer scientist who independently discovered the notion of NP-completeness.
- Neither Cook nor Levin were aware of the other's work due to the Iron Curtain.
- The fact that boolean satisfiability is NP-complete is now known as the Cook-Levin Theorem.

Pic: Wikipedia