All-pairs shortest paths

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Assume for simplicity that the input graph has no negative-weight cycles.

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- In the Floyd-Warshall algorithm, we assume we are given access to a graph G with n vertices as a $n \times n$ adjacency matrix W. The weights of the edges in G are represented as follows:

$$W_{ij} = \begin{cases} 0 & \text{if } i = j \\ \text{the weight of the edge } i \to j & \text{if such an edge exists} \\ \infty & \text{otherwise.} \end{cases}$$

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- In the case $k = 0$, $d_{ij}^{(0)} = W_{ij}$.
- On the other hand, for $k = n$, $d_{ij}^{(n)} = \delta(i, j)$.

A dynamic-programming recurrence

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We therefore have the following recurrence for $d_{ij}^{(k)}$:

$$d_{ij}^{(k)} = \begin{cases} W_{ij} & \text{if } k = 0 \\ \min\left\{ d_{ij}^{(k-1)}, d_{ik}^{(k-1)} + d_{kj}^{(k-1)} \right\} & \text{if } k \geq 1. \end{cases}$$

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1. $d^{(0)} \leftarrow W$
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3. for $i = 1$ to n
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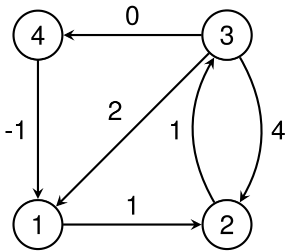
- The time complexity is clearly $O(n^3)$ and the algorithm is very simple.
- Correctness follows from the argument on the previous slide.

Example

Consider the following graph and its corresponding adjacency matrix:



$$\begin{pmatrix} 0 & 1 & \infty & \infty \\ \infty & 0 & 1 & \infty \\ 2 & 4 & 0 & 0 \\ -1 & \infty & \infty & 0 \end{pmatrix}$$

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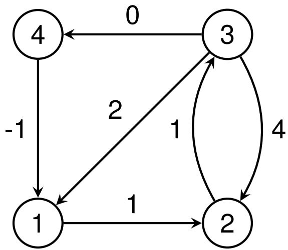

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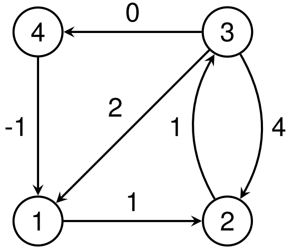

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$$d^{(1)} = \begin{pmatrix} 0 & 1 & \infty & \infty \\ \infty & 0 & 1 & \infty \\ 2 & 3 & 0 & 0 \\ -1 & 0 & \infty & 0 \end{pmatrix}, \quad d^{(2)} = \begin{pmatrix} 0 & 1 & 2 & \infty \\ \infty & 0 & 1 & \infty \\ 2 & 3 & 0 & 0 \\ -1 & 0 & 1 & 0 \end{pmatrix}$$

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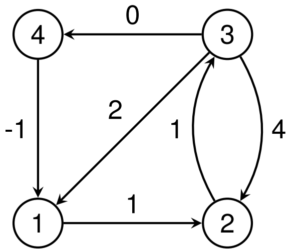

$$\begin{pmatrix} 0 & 1 & \infty & \infty \\ \infty & 0 & 1 & \infty \\ 2 & 4 & 0 & 0 \\ -1 & \infty & \infty & 0 \end{pmatrix}$$

$$d^{(3)} = \begin{pmatrix} 0 & 1 & 2 & {\color{blue}2} \\ {\color{blue}3} & 0 & 1 & {\color{blue}1} \\ 2 & 3 & 0 & 0 \\ -1 & 0 & 1 & 0 \end{pmatrix}$$

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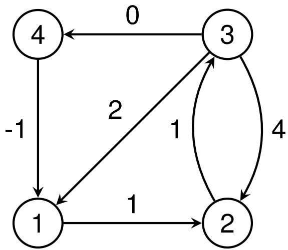

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- Then, for $k = 0$,

$$\Pi_{ij}^{(0)} = \begin{cases} \text{nil} & \text{if } i = j \text{ or } W_{ij} = \infty \\ i & \text{if } i \neq j \text{ and } W_{ij} \neq \infty. \end{cases}$$

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- For $k \geq 1$, we have essentially the same recurrence as for $d^{(k)}$. Formally,

$$\Pi_{ij}^{(k)} = \begin{cases} \Pi_{ij}^{(k-1)} & \text{if } d_{ij}^{(k-1)} \leq d_{ik}^{(k-1)} + d_{kj}^{(k-1)} \\ \Pi_{kj}^{(k-1)} & \text{otherwise.} \end{cases}$$

The Floyd-Warshall algorithm with predecessors

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for $k = 1$ to n
    for $i = 1$ to n
        for $j = 1$ to n
            if $d_{ij}^{(k-1)} \leq d_{ik}^{(k-1)} + d_{kj}^{(k-1)}$
                $d_{ij}^{(k)} \leftarrow d_{ij}^{(k-1)}$
                $\Pi_{ij}^{(k)} \leftarrow \Pi_{ij}^{(k-1)}$
            else
                $d_{ij}^{(k)} \leftarrow d_{ik}^{(k-1)} + d_{kj}^{(k-1)}$
                $\Pi_{ij}^{(k)} \leftarrow \Pi_{kj}^{(k-1)}$
return $d^{(n)}$.

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- For sparse graphs, its complexity $O(VE + V^2 \log V)$ (the same as Dijkstra) is faster than the Floyd-Warshall algorithm.
- We assume that we are given G as an adjacency list, and have access to a weight function $w(u, v)$ which tells us the weight of the edge $u \to v$.

Claim

For any edge $u \to v$, define

$$\widehat{w}(u,v) := w(u,v) + h(u) - h(v),$$

where h is an arbitrary function mapping vertices to real numbers. Then any path $p = v_0, \dots, v_k$ is a shortest path from v_0 to v_k with respect to the weight function $\widehat{w}$ if and only if it is a shortest path with respect to the weight function w.

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Proof

The total weights of p under $\widehat{w}$ and w are closely related:

$$\sum_{i=1}^{k} \widehat{w}(v_{i-1}, v_i) \quad = \quad \sum_{i=1}^{k} w(v_{i-1}, v_i) + h(v_{i-1}) - h(v_i)$$

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- So c is negative-weight under $\widehat{w}$ if and only if it is negative-weight under w.

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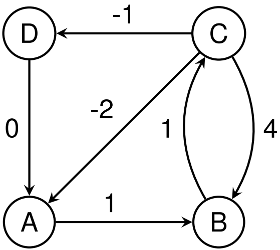
  for all edges $u \to v$.
- Then, if $\widehat{\delta}(u, v)$ is the weight of a shortest path from u to v with weight function $\widehat{w}$, $\delta(u, v) = \widehat{\delta}(u, v) + h(v) - h(u)$.

Example

Imagine we want to reweight the following graph:



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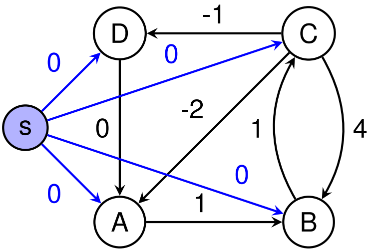

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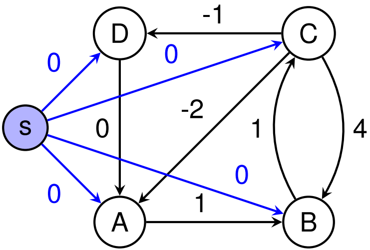

- Using Bellman-Ford, we compute

$$h(A) = -2, \quad h(B) = -1, \quad h(C) = 0, \quad h(D) = -1.$$

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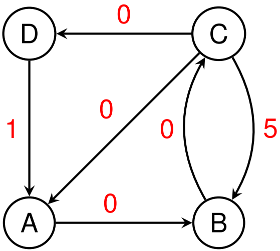
Reweighting according to *h* gives the following graph:



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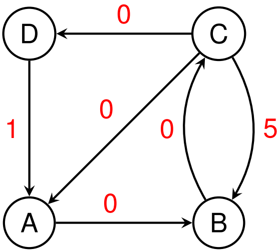

- For each pair of vertices u, v, $\delta(u,v) = \widehat{\delta}(u,v) + h(v) - h(u)$.
- For example, $\delta(C,A) = 0 - 2 - 0 = -2$ as expected.

Johnson's algorithm

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Johnson(G)

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4. $\quad\quad \widehat{w}(u, v) \leftarrow w(u, v) + \delta(s, u) - \delta(s, v)$
5. for each vertex $u \in G$
6. $\quad\quad$ compute $\widehat{\delta}(u, v)$ for all v using Dijkstra
7. $\quad\quad$ for each vertex $v \in G$
8. $\quad\quad\quad\quad d_{uv} \leftarrow \widehat{\delta}(u, v) + \delta(s, v) - \delta(s, u)$
9. return d

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- This can be significantly smaller than the runtime of the Floyd-Warshall algorithm if the input graph is sparse.

Shortest path algorithms: the summary

To compute single-source shortest paths in a directed graph G which is...

- unweighted: use breadth-first search in time $O(V + E)$;
- weighted with non-negative weights: use Dijkstra's algorithm in time $O(E + V \log V)$;
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To compute all-pairs shortest paths in a directed graph G which is...

- unweighted: use breadth-first search from each vertex in time $O(VE + V^2)$;
- weighted with non-negative weights: use Dijkstra's algorithm from each vertex in time $O(VE + V^2 \log V)$;
- weighted with negative weights: use Johnson's algorithm in time $O(VE + V^2 \log V)$.

Further Reading

- Introduction to Algorithms
  T.H. Cormen, C.E. Leiserson, R.L. Rivest and C. Stein.
  MIT Press/McGraw-Hill, ISBN: 0-262-03293-7.
  - Chapter 25 – All-Pairs Shortest Paths

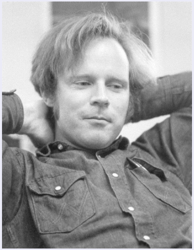
- Algorithms lecture notes, University of Illinois
  Jeff Erickson
  `http://www.cs.uiuc.edu/~jeffe/teaching/algorithms/`
  - Lecture 20 – All-pairs shortest paths

Biographical notes

The Floyd-Warshall algorithm was invented independently by Floyd and Warshall (and also Bernard Roy).

Robert W. Floyd (1936–2001)

- American computer scientist who did major work on compilers and initiated the field of programming language semantics.
- He completed his first degree (in liberal arts) at the age of 17 and won the Turing Award in 1978.
- Had his middle name legally changed to "W".



Pic: IEEE

Biographical notes

Stephen Warshall (1935–2006)

- Another American computer scientist whose other work included operating systems and compiler design.
- Supposedly he and a colleague bet a bottle of rum on who could first prove correctness of his algorithm.
- Warshall found his proof overnight and won the bet (and the rum).

Donald B. Johnson (d. 1994)

- Yet another American computer scientist. Founded the computer science department at Dartmouth College and invented the *d*-ary heap.