The Church-Turing thesis in a quantum world

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Introduction

Quantum complexity theory [Bernstein and Vazirani '97]

Just as the theory of computability has its foundations in the Church-Turing thesis, computational complexity rests on a modern strengthening of this thesis, which asserts that any "reasonable" model of computation can be efficiently simulated on a probabilistic Turing machine...

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However, the Turing Machine fails to capture all physically realizable computing devices for a fundamental reason: the Turing Machine is based on a classical physics model of the Universe, whereas current physical theory asserts that the Universe is quantum physical.

What does this imply for the Church-Turing thesis?

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However, there are certain quantum computations which we don't know how to simulate classically without exponential slowdown.

- The canonical example is factoring: Shor's quantum algorithm factorises an n-digit integer in time poly(n), but the best known classical algorithm takes time super-polynomial in n.
- So quantum computers pose a significant challenge to the strong Church-Turing thesis.

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- Models of computation where quantum computers provably outperform classical computers;

- How quantum computation helps us understand classical complexity theory.

Simulating physical systems

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- We are given a description of a system, and would like to determine something about its state at time t.

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- According to the laws of quantum mechanics, time evolution of the state $|\psi\rangle$ of a quantum system is governed by Schrödinger's equation,

$$i\hbar \frac{d}{dt}|\psi(t)\rangle = H(t)|\psi(t)\rangle,$$

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- Given H specifying a physical system, we would like to approximate the operator

$$U(t) = e^{-iHt}.$$

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- In general, H is too big to write down explicitly. If H describes a system of n particles (atoms, photons, ...), it has dimension exponential in n.

- However, with a quantum computer we can approximate $U(t)$ for the physically meaningful class of k-local Hamiltonians.

- These are Hamiltonians which are given by a sum of terms describing interactions between at most $k = O(1)$ particles. So H is described by a set of $O(1)$-dimensional matrices.

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4. Perform a measurement to extract information from $|\widetilde{\psi(t)}\rangle$.

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- We have access to x via an oracle which, given input i, returns the bit x_i. We allow the use of randomness and some probability of failure (e.g. up to $1/3$).
- For some functions f, clever strategies can allow us to compute $f(x)$ using far fewer than n queries.

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- However, it is easy to see that any classical algorithm requires $\Omega(n)$ queries.

Knowns and unknowns

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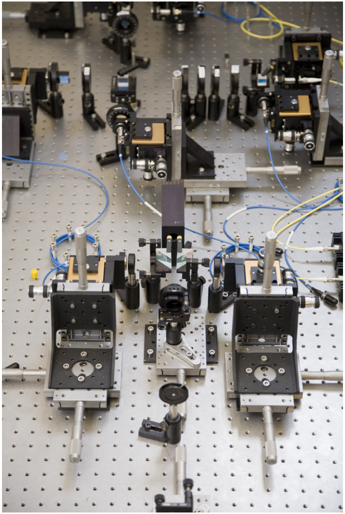
But there are still many open questions, such as:

- Can we achieve better than a quadratic separation for total functions?
- If the algorithm must succeed with certainty on all inputs, can we achieve better than a constant factor separation? (see [AM, Jozsa and Mitchison '11] for some examples of such separations).

A world without quantum computers?

- Small-scale quantum computers already exist in the lab.
- But what if we never manage to build large-scale quantum computers?
- Or what if quantum computers turn out to be easy to simulate classically?
- Studying quantum computing nevertheless has implications for the rest of computer science.



A computational hardness result

- Let T be a 3-index tensor, i.e. a $d \times d \times d$ array of complex numbers, such that $\sum_{i,j,k} |T_{ijk}|^2 = 1$.
- The injective tensor norm of T is defined as

$$\|T\|_{\text{inj}} := \max_{\substack{x,y,z, \\ \|x\|=\|y\|=\|z\|=1}} \left| \sum_{i,j,k=1}^{d} T_{ijk} x_i y_j z_k \right| .$$

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Theorem [Harrow & AM '11]

Assume that the (NP-complete) problem 3-SAT on n clauses can't be solved in time subexponential in n. Then there are universal constants $0 < s < c < 1$ such that distinguishing between $\|T\|_{\text{inj}} \leqslant s$ and $\|T\|_{\text{inj}} \geqslant c$ can't be done in time $\text{poly}(d)$.

Many other problems in tensor optimisation reduce to computing injective tensor norms.

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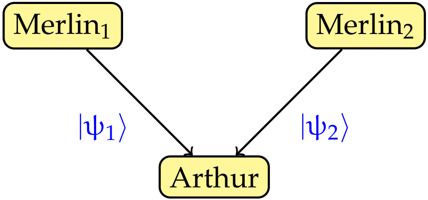

- Arthur has a hard decision problem to solve and has access to two separate provers ("Merlins"), who are all-powerful but cannot be trusted.
- The Merlins want to convince Arthur that the answer to the problem is "yes". Each of them sends Arthur a quantum state ("proof"). He then runs a quantum algorithm to check the proofs.

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- And it turns out that the maximal probability with which the Merlins can convince Arthur to output "yes" is given by the injective tensor norm of a tensor T.

- So, if we could compute $\|T\|_{\mathrm{inj}}$ up to an additive constant in time $\mathrm{poly}(d)$, we would have a subexponential-time algorithm for 3-SAT!

Other classical results with quantum proofs

Some other purely classical problems have quantum solutions.

- Classical communication complexity of the inner product function
- Lower bounds on locally decodable codes
- Rigidity of Hadamard matrices
- Finding low-degree approximating polynomials
- Closure properties of complexity classes
- …

For many more, see the survey "Quantum proofs for classical theorems" [Drucker and de Wolf '09].

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Thanks!