Three quantum learning algorithms

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Talk based on joint work with Andris Ambainis and ongoing joint work with Scott Aaronson, David Chen, Daniel Gottesman and Vincent Liew.

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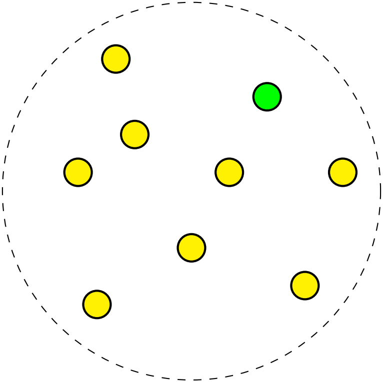
What is learning?



In this talk

Learning a set $S \equiv$ identifying an arbitrary, unknown object picked from S.

This talk

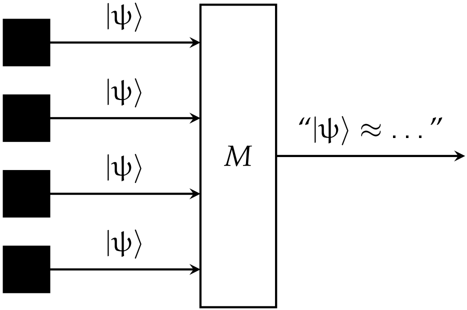
> “A little learning is a dangerous thing;
> drink deep, or taste not the Pierian spring:
> there shallow draughts intoxicate the brain,
> and drinking largely sobers us again.”
>
> — Alexander Pope

On this principle, I'll talk about three optimal quantum algorithms for learning an unknown...

- ...stabilizer state;
- ...low-degree multilinear polynomial;
- ...bit-string given access to “wildcard” queries.

Learning quantum states

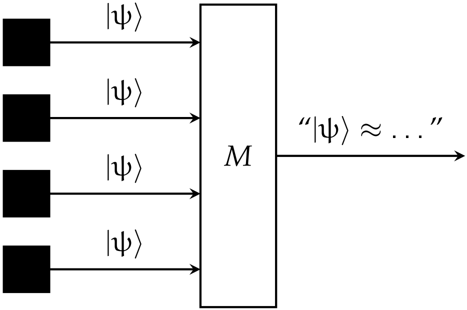
Consider the basic task of quantum state estimation.



- Given the ability to produce copies of an unknown n-qubit quantum state $|\psi\rangle$, we would like to estimate $|\psi\rangle$.
- Standard quantum state tomography uses $2^{\Theta(n)}$ copies of $|\psi\rangle$ to achieve constant fidelity.
- Can we do better?

Learning quantum states

Consider the basic task of quantum state estimation.



- To achieve constant fidelity between our guess and $|\psi\rangle$, we need $2^{\Omega(n)}$ copies of $|\psi\rangle$ [Bruss and Macchiavello '98].
- In order to determine $|\psi\rangle$ efficiently (using $\text{poly}(n)$ measurements) we must restrict to classes of states which have efficient descriptions, or change the problem.

Learning quantum states

Some examples where this has been achieved:

- [Cramer et al '11] give an efficient algorithm for learning matrix product states.
- [Aaronson '06] introduces "pretty good tomography": relax to attempting to predict the outcomes of "most" measurements on the state.
- [Flammia and Liu '11] and [da Silva et al '11] give efficient algorithms for certifying the production of certain states.

Learning stabilizer states

Today I'll talk about a learning algorithm for another important class of quantum states with efficient descriptions: stabilizer states.

- $|\psi\rangle$ is a stabilizer state of n qubits if there exists a subgroup G of 2^n pairwise commuting Pauli matrices (with phases) such that $M|\psi\rangle = |\psi\rangle$ for all $M \in G$.
- Examples include W states, cluster states, states occurring in quantum error-correcting codes, …

A stabilizer state of n qubits is completely specified by the identities of the elements of its stabilizer (n Pauli matrices on n qubits). There are $2^{\Theta(n^2)}$ stabilizer states of n qubits.

Learning stabilizer states

Theorem

There is a quantum algorithm which learns an unknown stabilizer state $|\psi\rangle$ given access to $O(n)$ copies of $|\psi\rangle$. The algorithm runs in time $O(n^3)$.

Notes on this result:

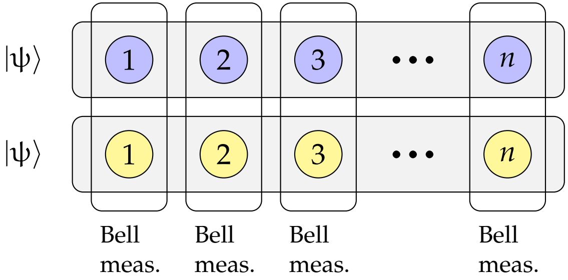
- By Holevo's theorem, this is optimal in terms of the scaling of the number of copies of $|\psi\rangle$ used.
- Any algorithm for learning stabilizer states requires $\Omega(n^2)$ time just to write the output.

The algorithm

The algorithm is based on the following subroutine.

Bell sampling

1. Create two copies of $|\psi\rangle$.
2. Measure each pair of qubits of $|\psi\rangle^{\otimes 2}$ in the Bell basis.



Learning stabilizer states

- For $z, x \in \{0,1\}$, write $\sigma_{zx} := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}^z \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^x$.
- For $s \in \{0,1\}^{2n}$, write

$$\sigma_s := \sigma_{s_1 s_2} \otimes \cdots \otimes \sigma_{s_{2n-1} s_{2n}}.$$

Fact

Let $|\psi\rangle$ be a state of n qubits. Performing Bell sampling on $|\psi\rangle^{\otimes 2}$ returns outcome s with probability

$$p_\psi(s) := \frac{|\langle\psi|\sigma_s|\psi^*\rangle|^2}{2^n}.$$

Bell sampling and stabilizer states

- Up to an overall phase every stabilizer state $|\psi\rangle$ can be written in the form

$$|\psi\rangle = \frac{1}{\sqrt{|A|}} \sum_{x \in A} i^{\ell(x)} (-1)^{q(x)} |x\rangle,$$

  where A is an affine subspace of $\mathbb{F}_2^n$, and $\ell, q : \{0,1\}^n \to \{0,1\}$ are linear and quadratic (respectively) polynomials over $\mathbb{F}_2$ [Dehaene and Moor '02].

- As ℓ is linear, $\ell(x) = s \cdot x$ for some $s \in \{0,1\}^n$.

- So $(-1)^{\ell(x)} = \prod_{i \in S} (-1)^{x_i}$ for some $S \subseteq [n]$.

- Hence

$$|\psi^*\rangle = \sigma_{10}^{\otimes S} |\psi\rangle.$$

Bell sampling and stabilizer states

- If we perform Bell sampling on $|\psi\rangle^{\otimes 2}$, we receive outcome t with probability

$$\frac{|\langle\psi|\sigma_t|\psi^*\rangle|^2}{2^n} = \frac{|\langle\psi|\sigma_t \sigma_{10}^{\otimes S}|\psi\rangle|^2}{2^n}.$$

- Let G stabilize $|\psi\rangle$ and let T denote the set of strings $t \in \{0,1\}^{2n}$ such that $\sigma_t \in G$, up to a phase. Then T is an n-dimensional linear subspace of $\mathbb{F}_2^{2n}$.

- So Bell sampling gives an outcome r which is uniformly distributed on the set $\{t \oplus s : t \in T\}$ for some $s \in \{0,1\}^{2n}$.

Bell sampling and stabilizer states

- For any two such outcomes r_1, r_2, the sum $r_1 \oplus r_2$ is uniformly distributed in T.
  - In order to find a basis for T, we can therefore produce $k+1$ Bell samples $r_0, r_1, \ldots, r_k$ and consider the uniformly random elements of T given by $r_1 \oplus r_0, r_2 \oplus r_0, \ldots, r_k \oplus r_0$.
  - If the dimension of the subspace of $\mathbb{F}_2^{2n}$ spanned by these vectors is n, any basis of this subspace is a basis for T.
- Although T does not contain information about phases, determining T suffices to uniquely determine $|\psi\rangle$.
  - Once we have found a basis for T, we can measure $|\psi\rangle$ in the eigenbasis of each corresponding Pauli matrix M to decide whether $M|\psi\rangle = |\psi\rangle$ or $M|\psi\rangle = -|\psi\rangle$.

Learning stabilizer states

The algorithm

1. Set $S = \emptyset$.
2. Create two copies of $|\psi\rangle$ and perform Bell sampling, obtaining outcome r_0.
3. Repeat the following $2n$ times:
    1. Create two copies of $|\psi\rangle$ and perform Bell sampling, obtaining outcome r.
    2. Add $r \oplus r_0$ to S.
4. Determine a basis for S; call this basis B.
5. For each element of B, measure a copy of $|\psi\rangle$ in the eigenbasis of the corresponding Pauli matrix M to determine whether $M|\psi\rangle = |\psi\rangle$ or $M|\psi\rangle = -|\psi\rangle$.

Summary of learning stabilizer states

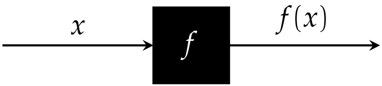
- The algorithm uses $O(n)$ copies of $|\psi\rangle$. Time complexity is dominated by finding a basis for S ($O(n^3)$ time or better).

- The algorithm fails (i.e. does not identify $|\psi\rangle$) if each of the $2n$ samples $r \oplus r_0$ lies in a subspace of T of dimension at most $n-1$. This occurs with probability at most 2^{-n}.

- We also have an alternative algorithm which uses $\Theta(n^2)$ copies of $|\psi\rangle$ but only makes single-copy Pauli measurements.

Learning classical oracles

Consider the following purely classical problem.



- We are given access to a function $f : \{0,1\}^n \to \{0,1\}$. We would like to identify f.
- If f is arbitrary, we need 2^n queries (uses of f).
- If f is picked from a known set $\mathcal{F}$, we need at least $\log_2 |\mathcal{F}|$ queries.
- We say that $\mathcal{F}$ can be learned using t queries if any function $f \in \mathcal{F}$ can be identified with t uses of f (perhaps allowing some probability of error).

Learning classical oracles on a quantum computer

- On a quantum computer, we have the ability to query f in superposition, i.e. to perform the map

$$|x\rangle|z\rangle \mapsto |x\rangle|z + f(x)\rangle.$$

- One of the oldest results in quantum computing: the Bernstein-Vazirani algorithm [Bernstein and Vazirani '97].

Theorem (Bernstein and Vazirani)

The class of linear functions $f : \mathbb{F}_2^n \to \mathbb{F}_2$ can be learned with certainty using 1 quantum query.

f is linear if $f(x+y) = f(x) + f(y)$; equivalently, $f(x) = \ell \cdot x$ for some $\ell \in \mathbb{F}_2^n$.

Learning multilinear polynomials

$f : \mathbb{F}_q^n \to \mathbb{F}_q$ is a degree d multilinear polynomial:

$$f(x) = \sum_{S \subseteq [n], |S| \leqslant d} \alpha_S \prod_{i \in S} x_i$$

for some coefficients $\alpha_S \in \mathbb{F}_q$, where $[n] := \{1, \ldots, n\}$.

- Note that for $S = \emptyset$ we define $\prod_{i \in S} x_i = 1$.
- For example, any multilinear polynomial of degree 3 can be written as

$$f(x) = \alpha_\emptyset + \sum_i \alpha_{\{i\}} x_i + \sum_{i<j} \alpha_{\{i,j\}} x_i x_j + \sum_{i<j<k} \alpha_{\{i,j,k\}} x_i x_j x_k.$$

- In the important special case $q = 2$ (boolean functions), every polynomial is multilinear.
- The set of degree d polynomials over $\mathbb{F}_2$ are known as the binary Reed-Muller code of order d.

Learning multilinear polynomials

Fact

The class of degree d multilinear polynomials in n variables over $\mathbb{F}_q$ can be learned using $O(n^d)$ classical queries, and this is optimal.

- Upper bound: It suffices to query $f(x)$ for all strings $x \in \mathbb{F}_q^n$ that contain only 0 and 1, and such that $|x| \leqslant d$.
- Lower bound: there are $q^{\Theta(n^d)}$ distinct multilinear degree d polynomials of n variables over $\mathbb{F}_q$; each classical query to f only provides $\log_2 q$ bits of information.

Learning multilinear polynomials

Theorem

The class of degree d multilinear polynomials in n variables over $\mathbb{F}_q$ can be learned exactly using $O(n^{d-1})$ quantum queries, and this is optimal.

Notes:

- The lower bound follows from Holevo's theorem.
- The Bernstein-Vazirani algorithm is the case $q = 2$, $d = 1$.
- Rötteler previously gave a bounded-error quantum algorithm for the case $q = 2$, $d = 2$ [Rötteler '09].
- A quantum algorithm for estimating a quadratic form over the reals had previously been given by Jordan [Jordan '08].

The algorithm

We use the following lemma [de Beaudrap et al '02, van Dam et al '02].

Lemma 1

Let $f : \mathbb{F}_q^n \to \mathbb{F}_q$ be linear, and let $g : \mathbb{F}_q^n \to \mathbb{F}_q$ be the function $g(x) = f(x) + \beta$ for some constant $\beta \in \mathbb{F}_q$. Then f can be determined exactly using one quantum query to g.

- Proof: query f in superposition and use the QFT.

The algorithm

For $S \subseteq [n]$, $|S| = k$, define

$$f_S(x) = \sum_{\beta_1,\dots,\beta_k \in \{0,1\}} (-1)^{k-\sum_{i=1}^k \beta_i} f\left(x + \sum_{j=1}^{k} \beta_j e_{S_j}\right).$$

Here e_i is the i'th element in the standard basis for $\mathbb{F}_q^n$; the inner sum is over $\mathbb{F}_q^n$ and the outer sum is over $\mathbb{F}_q$.

- For example, if $S = \{1, 2\}$:

$$f_S(x) = f(x) - f(x + e_1) - f(x + e_2) + f(x + e_1 + e_2).$$

- A query to f_S can be simulated using 2^k queries to f.
- Define the discrete derivative of f in direction $i \in [n]$ as

$$(\Delta_i f)(x) := f(x + e_i) - f(x).$$

- Then $f_S(x) = (\Delta_{S_1} \Delta_{S_2} \dots \Delta_{S_k} f)(x)$.

The algorithm

We will be interested in querying f_S for sets S of size $d-1$. In this case, we have the following characterisation for multilinear polynomials f.

Lemma 2

Let $f : \mathbb{F}_q^n \to \mathbb{F}_q$ be a multilinear polynomial of degree d with expansion

$$f(x) = \sum_{T \subseteq [n], |T| \leqslant d} \alpha_T \prod_{i \in T} x_i.$$

Then, for any S such that $|S| = d-1$,

$$f_S(x) = \alpha_S + \sum_{k \notin S} \alpha_{S \cup \{k\}} x_k.$$

Proof: follows easily from expressing f in terms of discrete derivatives.

Learning all the degree d terms

The algorithm

foreach $S \subseteq [n]$ *such that* $|S| = d - 1$ **do**
  Use one query to f_S to learn $\alpha_{S \cup \{k\}}$, for all $k \notin S$;
end
Output the function $f_d(x) = \sum_{S \subseteq [n], |S| = d} \alpha_S \prod_{i \in S} x_i$;

Proof of correctness:

- By Lemma 2, for any S such that $|S| = d - 1$, knowledge of the degree 1 component of f_S is sufficient to determine $\alpha_{S \cup \{k\}}$ for all $k \notin S$.

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- So knowing the degree 1 part of f_S for all $S \subseteq [n]$ such that $|S| = d - 1$ is sufficient to completely determine all degree d coefficients of f.

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Proof of correctness:

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- So the algorithm completely determines the degree d component of f using $\binom{n}{d-1}$ queries to f_S, each of which uses 2^{d-1} queries to f.

Finishing up

- Once the degree d component of f has been learned, f can be reduced to a degree $d-1$ polynomial by crossing out the degree d part whenever the oracle for f is called.
- Whenever the oracle is called on x, we subtract $f_d(x)$ from the result (recall f_d is the degree d part of f), at no extra query cost.
- Inductively, f can be determined completely using

$$2^{d-1}\binom{n}{d-1} + 2^{d-2}\binom{n}{d-2} + \cdots + 2n + 1 + 1$$

  queries; the last query is to determine the constant term $\alpha_\emptyset$, which can be achieved by classically querying $f(0^n)$.
- The number of queries used is therefore $O(n^{d-1})$ for constant d.

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Theorem

Search with wildcards can be solved with $O(\sqrt{n})$ quantum queries on average.

Solving SWW

The solution to SWW is based on this claim:

Measurement Lemma

Fix $n \geqslant 1$ and, for any $0 \leqslant k \leqslant n$, set

$$|\psi_x^k\rangle := \frac{1}{\binom{n}{k}^{1/2}} \sum_{S \subseteq [n], |S|=k} |S\rangle |x_S\rangle,$$

where $|x_S\rangle := \bigotimes_{i \in S} |x_i\rangle$. Then, for any $k = n - O(\sqrt{n})$, there is a quantum measurement (POVM) which, on input $|\psi_x^k\rangle$, outputs $\widetilde{x}$ such that the expected Hamming distance $d(x, \widetilde{x})$ is $O(1)$.

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Why does this let us solve SWW?

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$$\sum_{S':S\subseteq[n],|S'|=k'} |S'\rangle \left(\sum_{S:S\subseteq S',|S|=k} |S\rangle|x_S\rangle \right) = \sum_{S:S\subseteq[n],|S|=k'} |S\rangle|\psi_{x_S}^k\rangle,$$

so if we can map $|\psi_{x_S}^k\rangle \mapsto |x_S\rangle$, we've made $|\psi_x^{k'}\rangle$.

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- After each measurement, an expected $O(1)$ bits are incorrect.
- How to fix these?

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In particular, we would like to minimise the dependence on n.

Classical results

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- Many applications known: molecular biology, data streaming algorithms, compressed sensing, pattern matching in strings, …

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  - Lower bound: information-theoretic argument.
  - Upper bound: (essentially) binary search.

- If we restrict to nonadaptive queries, the bound becomes essentially $\Theta(\min\{k^2 \log n, n\})$.

- Many applications known: molecular biology, data streaming algorithms, compressed sensing, pattern matching in strings, ...

- See the book "Combinatorial Group Testing and Its Applications" [Du and Hwang '00] for more.

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- In the CGT problem, we have access to an oracle which computes $f(s) = \bigvee_i x_i s_i$ for arbitrary $s \in \{0,1\}^n$. But if $|x| \leqslant 1$, then for any s, $\bigvee_i x_i s_i = x \cdot s$.

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3. Apply Hadamard gates to each qubit of the first register and measure to obtain x.

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- We can check whether the index $\tilde{i}$ we received really is a 1 by making one more query to index $\tilde{i}$.
- Following each successful query, we reduce k by 1 and exclude the bit that we just learned from future queries.
- In order to learn x completely, the expected overall number of queries used is $O(k)$.

Back to search with wildcards

- When we measure $|\psi_x^k\rangle$, we get an outcome $\widetilde{x}$ such that $d(\widetilde{x}, x) = O(1)$.
- We want to determine x, which is equivalent to determining $\widetilde{x} \oplus x$, a string of Hamming weight $O(1)$.
- A wildcard query corresponding to $S \subseteq [n]$ and $\widetilde{x}_S \oplus y$, $y \in \{0, 1\}^{|S|}$, returns 1 iff all bits of $\widetilde{x}_S$ are correct.
- So we can use the algorithm for CGT to find, and correct, all incorrect bits in $O(1)$ queries.

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We finally need to prove we can distinguish the $|\psi_x^k\rangle$ states. We use the pretty good measurement (PGM).

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The probability that the PGM outputs y on input $|\psi_x^k\rangle$ is precisely $(\sqrt{G})_{xy}^2$, where

$$G_{xy} = \langle\psi_x^k|\psi_y^k\rangle = \frac{1}{\binom{n}{k}} \sum_{S\subseteq[n],|S|=k} [x_S = y_S] = \frac{\binom{n-d(x,y)}{k}}{\binom{n}{k}}.$$

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- G_{xy} depends only on $x \oplus y$, so G is diagonalised by the Fourier transform over $\mathbb{Z}_2^n$ and D_k does not depend on x.
- D_k can be upper bounded using Fourier duality and some combinatorics.

Summary

We can learn…

- … n-qubit stabilizer states with $O(n)$ copies;
- … degree d n-variate multilinear polynomials with $O(n^{d-1})$ queries;
- … n-bit strings with $O(\sqrt{n})$ wildcard queries.

Open problems:

- Determine the quantum query complexity of CGT.
- Other applications of SWW! A possible example: testing juntas.

Thanks!

Some further reading:

- The algorithm for learning multilinear polynomials: **arXiv:1105.3310**
- The algorithm for search with wildcards: **arXiv:1210.1148** (joint work with Andris Ambainis)
- The algorithm for learning stabilizer states: **arXiv:13??.????** (joint work with Scott Aaronson, David Chen, Daniel Gottesman and Vincent Liew)