Injective tensor norms and open problems in quantum information

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Introduction

This talk is about how several interesting open problems in quantum information can be phrased in terms of injective tensor norms:

- Finding the pure quantum state which is most entangled with respect to the geometric measure of entanglement;
- Determining whether multiple-prover quantum Merlin-Arthur games obey a parallel repetition theorem;
- Deciding whether quantum query algorithms can be simulated by classical query algorithms on most inputs.

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- If T is a 2-index tensor (i.e. a matrix),

  $$\|T\|_p^{\text{inj}} = \|T\|_{p \to p'},$$

  where for any matrix M

  $$\|M\|_{p \to q} := \max_{v, \|v\|_p = 1} \|Mv\|_q.$$

  When $p = 2$ this is the operator norm $\|T\|_{\text{op}}$, i.e. the largest singular value of T.

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- Observe that trivially $0 \leqslant E_{\text{geom}}(|\psi\rangle) \leqslant n \log_2 d$, by writing $|\psi\rangle$ in an arbitrary product basis.

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- Application: Can be used to replace finding the ground-state energy of a local Hamiltonian (a QMA-hard problem) with an optimisation over product states (in the complexity class NP) [Gharibian and Kempe '11].
- But a very natural question in its own right! "What is the most entangled quantum state?"

Some (easy and well-known) partial results

Proposition

For any $|\psi\rangle \in B(\mathbb{C}^d \otimes \mathbb{C}^d)$, $E_{\mathrm{geom}}(|\psi\rangle) \leqslant \log_2 d$, which is achieved by

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- [Jung et al '08] show that this cannot be tight for $n > 2$.
- For any symmetric state $|\psi\rangle$, the (often much tighter) bound

$$E_{\text{geom}}(|\psi\rangle) \leqslant \log_2 \binom{n+d-1}{d-1} = O(d(\log n + \log d))$$

holds (e.g. see [Aulbach '11]).

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&= \frac{1}{d^{n-1}} \sum_{i_1=1}^{d} \sum_{i_2,\dots,i_n=1}^{d} |\psi_{i_1,\dots,i_n}|^2 = \frac{1}{d^{n-1}}.
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Pick $|\psi\rangle \in B((\mathbb{C}^d)^{\otimes n})$ at random (according to Haar measure). Then with high probability

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- No known candidate for an explicit quantum state which beats this bound!

From injective tensor norms to quantum Merlin-Arthur games

- A separable state $\rho \in \mathrm{SEP} \subset \mathcal{B}(\mathbb{C}^d \otimes \mathbb{C}^d)$ is a state of the form
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- Define the support function of the separable states,

$$\begin{aligned} h_{\mathrm{SEP}}(M) &:= \max_{\rho \in \mathrm{SEP}} \operatorname{tr} M\rho \\ &= \max_{|\phi_1\rangle, |\phi_2\rangle \in B(\mathbb{C}^d)} \langle \phi_1 | \langle \phi_2 | M | \phi_1 \rangle | \phi_2 \rangle \end{aligned}$$

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- It turns out that h_{SEP} can be expressed in terms of injective tensor norms.

h_{SEP} and injective tensor norms

Let $T_{i,j,k}$ be an arbitrary 3-index tensor. Then

$$(\|T\|_2^{\text{inj}})^2 = \max_{x,y,z \in B(\mathbb{C}^d)} \left| \sum_{i,j,k=1}^{d} T_{i,j,k} x_i y_j z_k \right|^2$$

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Quantum Merlin-Arthur games

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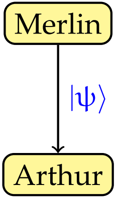

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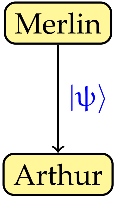

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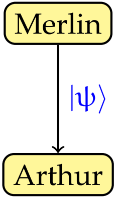

- Arthur has some decision problem of size n to solve, and Merlin wants to convince him that the answer is “yes”.
- Merlin sends him a quantum state $|\psi\rangle$ of poly(n) qubits. Arthur runs some polynomial-time quantum algorithm $\mathcal{A}$ on $|\psi\rangle$ and his input and outputs “yes” if the algorithm says “accept”.

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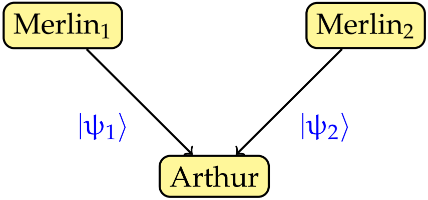

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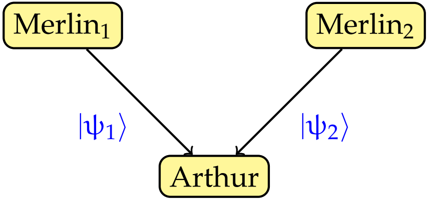

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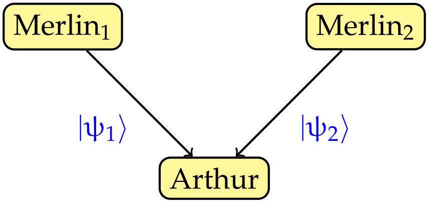

- This might be more powerful than QMA because the lack of entanglement helps Arthur tell when the Merlins are cheating.
- For example, 3-SAT on n clauses can be solved by a QMA(2) protocol with constant probability of error using proofs of length $O(\sqrt{n}\,\mathrm{polylog}(n))$ qubits [Harrow and AM '10].

QMA(2) and h_{SEP}

Fact

For a given "no" problem instance, let Arthur's measurement operator corresponding to a "yes" outcome be M. Then the maximal probability with which the Merlins can force Arthur to incorrectly output "yes" is precisely $h_{\mathrm{SEP}}(M)$.

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- Via the connection to 3-SAT, implies computational hardness of approximating $h_{\mathrm{SEP}}(M)$.
- Unless there exists a subexponential-time algorithm for 3-SAT, there is no polynomial-time algorithm for estimating $h_{\mathrm{SEP}}(M)$ up to an additive constant.

Multiplicativity of h_{SEP}

Open problem 2

Is h_{SEP} weakly multiplicative? i.e. does it hold that, for all M,

$$h_{\text{SEP}}(M^{\otimes n}) \leqslant h_{\text{SEP}}(M)^{\alpha n}$$

for some $0 < \alpha < 1$?

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- If true, this would imply that QMA(2) protocols obey a form of parallel repetition: to achieve exponentially small failure probability, Arthur can simply repeat the protocol n times in parallel.
- There are also connections to many other open additivity/multiplicativity problems in quantum information theory via a link to maximum output p-norms of quantum channels.

Some known partial results

Theorem [Werner and Holevo '02], [Grudka et al '09]

There exists M such that

$$h_{\mathrm{SEP}}(M^{\otimes 2}) = h_{\mathrm{SEP}}(M)(1 - o(1)).$$

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- This result implies that strict parallel repetition does not hold for QMA(2) protocols.
- Connected to the failure of the famous additivity conjecture for Holevo capacity of quantum channels [Hastings '09].

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Theorem [AM '11]

Pick the subspace onto which M projects at random (according to Haar measure) from the set of all dimension r subspaces of $\mathbb{C}^{d_A} \otimes \mathbb{C}^{d_B}$. Then the probability that $h_{\text{SEP}}(M)$ is *not* weakly multiplicative with exponent $1/2 - o(1)$ is exponentially small in $\min\{r, d_A, d_B\}$.

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Note: The above result holds with the following (fairly weak) restrictions on r, d_A, d_B:

- $r = o(d_A d_B)$.
- $\min\{r, d_A, d_B\} \geqslant 2(\log_2 \max\{d_A, d_B\})^{3/2}$.

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The proof uses ideas from random matrix theory.

Simulation of quantum query algorithms

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- On the other hand, for any total function f, there can be at most a polynomial separation between quantum and classical query complexity [Beals et al '01].
- Raises the natural question: how strict does the promise on the input have to be in order to get an exponential speed-up?

Quantum queries and injective tensor norms

Conjecture A [Aaronson and Ambainis '09]

Let Q be a quantum algorithm which makes T queries to x. Then, for any $\epsilon > 0$, there is a classical algorithm which makes $\text{poly}(T, 1/\epsilon, 1/\delta)$ queries to x, and approximates Q's success probability to within $\pm\epsilon$ on a $1 - \delta$ fraction of inputs.

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- Given known results, essentially the strongest conjecture one could make about classical simulation of quantum query algorithms.
- Aaronson and Ambainis show that Conjecture A follows from the following, more mathematical conjecture...

Quantum queries and injective tensor norms

Conjecture B [Aaronson and Ambainis '09, slightly modified]

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a degree d multivariate polynomial such that $|f(x)| \leqslant 1$ for all $x \in \{\pm 1\}^n$ and $\mathrm{Var}(f) \geqslant \epsilon$. Then there exists $j \in \{1, \ldots, n\}$ such that

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In this conjecture:

$$\begin{aligned}
\mathrm{Var}(f) &= \mathbb{E}_x[(f(x) - \mathbb{E}[f])^2] = \frac{1}{2^n} \sum_{x \in \{\pm 1\}^n} \left(f(x) - \frac{1}{2^n} \sum_{y \in \{\pm 1\}^n} f(x) \right)^2 \\
\mathrm{Inf}_j(f) &= \frac{1}{2^{n+2}} \sum_{x \in \{\pm 1\}^n} (f(x) - f(x^j))^2
\end{aligned}$$

A very special case of this conjecture

- Let $f : (\mathbb{R}^s)^t \to \mathbb{R}$ be the multilinear form corresponding to a tensor $T \in (\mathbb{R}^s)^t$.

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- The influence of variable (j,k) on f is

$$\mathrm{Inf}_{(j,k)}(f) = \sum_{i_1,\ldots,i_{j-1},i_{j+1},\ldots,i_t} T^2_{i_1,\ldots,i_{j-1},k,i_{j+1},\ldots,i_t}.$$

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Open problem 3

Assume that $\|T\|_\infty^{\mathrm{inj}} \leqslant 1$. Show that, for all $1 \leqslant j \leqslant t$,

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This would imply Conjecture B of Aaronson and Ambainis for the special case where f is a multilinear form.

Open problem 3 implies a special case of Conjecture B

- First observe that $\|T\|_{\infty}^{\text{inj}} \leqslant 1$ is equivalent to $|f(x)| \leqslant 1$ for $x \in \{\pm 1\}^{st}$.

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$$\mathrm{Var}(f) \quad \leqslant \quad \sum_{j,k} \mathrm{Inf}_{(j,k)}(f) \leqslant \max_{j,k} \mathrm{Inf}_{(j,k)}(f)^{1/2} \sum_{j,k} \mathrm{Inf}_{(j,k)}(f)^{1/2}$$

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$$\begin{aligned}\mathrm{Var}(f) &\leqslant \sum_{j,k} \mathrm{Inf}_{(j,k)}(f) \leqslant \max_{j,k} \mathrm{Inf}_{(j,k)}(f)^{1/2} \sum_{j,k} \mathrm{Inf}_{(j,k)}(f)^{1/2} \\ &\leqslant \mathrm{poly}(t) \max_{j,k} \mathrm{Inf}_{(j,k)}(f)^{1/2},\end{aligned}$$

so

$$\max_{j,k} \mathrm{Inf}_{(j,k)}(f) \geqslant \mathrm{poly}(\mathrm{Var}(f)/t).$$

Partial results

Theorem [Bohnenblust and Hille '31]

Assume that $\|T\|_\infty^{\text{inj}} \leqslant 1$. Then there is a universal constant $C > 1$ such that, for all $1 \leqslant j \leqslant t$,

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- This is a generalisation of Littlewood's 4/3 inequality [Littlewood '30].
- The constant C has gradually been improved over the years...

Partial results

Theorem [AM '11, folklore?]

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a symmetric degree d multivariate polynomial such that $|f(x)| \leqslant 1$ for all $x \in \{\pm 1\}^n$ and $\mathrm{Var}(f) \geqslant \epsilon$. Then, for all $j \in \{1, \ldots, n\}$,

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- A symmetric polynomial $f(x)$ depends only on the Hamming weight of $x \in \{\pm 1\}^n$, i.e. the number of 1s in x.
- For such polynomials, all influences are equal.

Conclusions

- Injective tensor norms are a powerful general framework in which to attack many open problems in quantum information theory.
- Many of these problems are accessible and can be stated purely mathematically, with no reference to quantum information.
- This doesn’t stop them from probably being very hard!

Thanks!

Further reading:

- "Classification of Entanglement in Symmetric States" [Aulbach '11] – an entire PhD thesis on the geometric measure of entanglement (!)
- "An efficient test for product states, with applications to quantum Merlin-Arthur games" [Harrow and AM '10] (arXiv:1001.0017) – stay tuned for a new version giving many other interpretations of $h_{\mathrm{SEP}}(M)$
- "Weak multiplicativity for random quantum channels" [AM '11] (arXiv:1112.5271) – includes references to many other papers on multiplicativity questions
- "The role of structure in quantum speed-ups" [Aaronson and Ambainis '09].

Conjecture B implies Conjecture A (sketch)

Consider the following algorithm:

1. If $\mathrm{Var}(f) \leqslant (\delta\epsilon)^2$, stop and return $\mathbb{E}_x[f(x)]$.
2. Query the variable j such that $\mathrm{Inf}_j(f)$ is maximal and set f to be the resulting function.
3. Go to step 1.

Theorem [Aaronson and Ambainis '09]

Assuming Conjecture B, this algorithm terminates in expected time $\mathrm{poly}(d, 1/\epsilon, 1/\delta)$, where the expectation is taken over x, and computes $f(x)$ to within ϵ on at least a $1-\delta$ fraction of inputs x.

Conjecture B implies Conjecture A (sketch)

- Let $\tilde{f}$ be the function computed by the algorithm (observe that it always terminates).
- We have

$$\Pr_x[|f(x) - \tilde{f}(x)| \geqslant \epsilon] \leqslant \frac{\mathbb{E}_x[|f(x) - \tilde{f}(x)|]}{\epsilon} \leqslant \frac{\mathrm{Var}(f)^{1/2}}{\epsilon} \leqslant \delta.$$

- The algorithm terminates when $\mathrm{Var}(f) \leqslant (\delta\epsilon)^2$, and at the beginning of the algorithm $\mathrm{Var}(f) \leqslant \sum_j \mathrm{Inf}_j(f) \leqslant d$.
- The expected decrease in the total influence with each query is $\max_j \mathrm{Inf}_j(f)$.
- Assuming Conjecture B, this is lower bounded by $\mathrm{poly}(\mathrm{Var}(f)/d) \geqslant \mathrm{poly}(\delta\epsilon/d)$.
- Thus the expected number of queries until the algorithm terminates is at most $\mathrm{poly}(d, 1/\epsilon, 1/\delta)$.