On the dimension of subspaces with bounded Schmidt rank

Toby Cubitt,[1] Ashley Montanaro,[2] and Andreas Winter[1,3]
[1] *Department of Mathematics, University of Bristol, University Walk, Bristol BS8 1TW, UK*
[2] *Department of Computer Science, University of Bristol, Woodland Road, Bristol, BS8 1UB, UK*
[3] *Quantum Information Technology Lab, National University of Singapore, 2 Science Drive 3, Singapore 117542*
(Dated: 5 June 2007)

We consider the question of how large a subspace of a given bipartite quantum system can be when the subspace contains only highly entangled states. This is motivated in part by results of Hayden *et al.*, which show that in large $d \times d$–dimensional systems there exist random subspaces of dimension almost d^2, all of whose states have entropy of entanglement at least $\log d - O(1)$. It is also a generalisation of results on the dimension of completely entangled subspaces, which have connections with the construction of unextendible product bases. Here we take as entanglement measure the Schmidt rank, and determine, for every pair of local dimensions d_A and d_B, and every r, the largest dimension of a subspace consisting only of entangled states of Schmidt rank r or larger. This exact answer is a significant improvement on the best bounds that can be obtained using random subspace techniques. We also determine the converse: the largest dimension of a subspace with an *upper* bound on the Schmidt rank. Finally, we discuss the question of subspaces containing only states with Schmidt *equal to r*.

Introduction. Entanglement is at the heart of quantum information theory, and this property of quantum systems is ultimately responsible for new information tasks such as teleportation [1], quantum key agreement [2, 3] or quantum computational speedup [4]. Consequently, a theory of measuring and comparing the entanglement content of quantum states has emerged [5], which attempts to classify states according to their non-classical capabilities. It is, however, remarkable how large a number of entanglement measures have been put forward [5, 6], indicating that the structure of entanglement is not one that can be captured by a single number. One particular measure is the Schmidt rank of a pure bipartite state $|\psi\rangle$, i.e. the number of non-zero coefficients λ_i in the — essentially unique — Schmidt form:

$$|\psi\rangle = \sum_i \lambda_i |e_i\rangle_A |f_i\rangle_B.$$

This measure has even been extended to mixed states, as the maximum Schmidt rank in an optimal pure state decomposition [7], but the convex hull construction could also be considered. For pure states, the Schmidt rank is indeed the unique invariant under the class of stochastic local operations and classical communication (SLOCC).

Here we ask, and answer completely, the question: what is the maximum dimension of a subspace S in a $d_A \times d_B$ bipartite system such that every state in S has Schmidt rank at least r? This is trivial for $r = 1$, so we can assume $r \geq 2$; also, the Schmidt rank can be at most $\min\{d_A, d_B\}$, which we will assume without loss of generality to be d_A.

There are two extreme cases. The first, $r = 2$ (i.e. a subspace that contains no product state), is addressed in Refs. [8, 9]; the answer is $(d_A - 1)(d_B - 1)$. The other, $r = d_A = d_B =: d$, has an elementary solution: the answer is 1 (take any one-dimensional subspace spanned by a vector of maximum Schmidt rank d). To show this, consider any two-dimensional subspace spanned by unit vectors $|\varphi\rangle, |\psi\rangle \in \mathbb{C}^d \otimes \mathbb{C}^d$. We want to show that at least one superposition $|\phi_x\rangle = |\varphi\rangle + x\,|\psi\rangle$ has Schmidt rank less than d. The crucial observation is that we can arrange the coefficients of a state vector $|\phi\rangle$ in the computational basis $\{|i\rangle\,|j\rangle\}_{i,j=1,\dots,d}$, into a $d \times d$ matrix $M(\phi)$, and that the Schmidt rank of the state vector equals the linear rank of the associated matrix. In other words, the statement that $|\phi_x\rangle$ has Schmidt rank less than r is captured by the vanishing of the determinant $\det M(\phi_x)$. But the latter is a non-constant polynomial in x of degree $\leq d$. Hence, it must have a root in the complex field, and the corresponding $|\phi_x\rangle$ has Schmidt rank $r - 1$ or less.

It turns out that the generalisation to arbitrary r rests on the same matrix representation, and the characterisation of Schmidt rank via vanishing of certain determinants again plays a crucial role. It involves, however, much deeper algebraic geometry machinery, extending the above use of the fundamental theorem of algebra. This has been considered in many different contexts in the algebraic geometry literature, e.g. [10, 11]. In the following, we give a more elementary treatment of the results that lead to an upper bound on the subspace dimension. We then give an explicit construction of such a subspace, for any d_A, d_B and r, which saturates this bound. As far as we are aware, this explicit construction is new, and this, rather than the dimension formula itself, turns out to be the key to applying these results to the infamous additivity problem in quantum information theory [12].

Notation and Terminology. We will denote the Schmidt rank of a bipartite pure state $|\psi\rangle_{AB}$ (nonzero, but not necessarily normalised) by $\mathrm{Sch}(|\psi\rangle_{AB})$.

We say that a subspace S of the bipartite space in question has Schmidt rank $\geq r$ if all its non-zero vectors have Schmidt rank $\geq r$ (analogously for $\leq r$ and $= r$). If

M is a matrix, R is a set of indices for rows of M, and C is a set of indices for columns, then $M_{\{R,C\}}$ denotes the submatrix formed by deleting all rows and columns *other than* those in R and C.

The projective space of dimension d is denoted by $\mathbb{P}^d$. This is the space of lines (one-dimensional subspaces) of $\mathbb{C}^{d+1}$; i.e. it is obtained from the nonzero elements of $\mathbb{C}^{d+1}$ by identifying collinear vectors. If a subset $S \subseteq \mathbb{C}^{d+1}$ is a union of lines, this identification associates to it a natural projectification of S, denoted by $\mathbb{P}(S) \subseteq \mathbb{P}^d$; see e.g. [13–16] for this and related notions from algebraic geometry.

For a state vector $|\psi\rangle \in \mathbb{C}^{d_A} \otimes \mathbb{C}^{d_B}$, with fixed local bases of the two Hilbert spaces, such that $|\psi\rangle = \sum_{i,j} c_{ij} |i\rangle |j\rangle$, define the $d_A \times d_B$ matrix $M(\psi) = (c_{ij})_{i=1,\dots,d_A, j=1,\dots,d_B}$. This identifies $\mathbb{C}^{d_A} \otimes \mathbb{C}^{d_B}$ with the space $\mathbb{M}(d_A, d_B)$ of $d_A \times d_B$ matrices.

Bounding the dimension of highly entangled subspaces. We first state some preparatory lemmas relating bipartite states to matrices, whose proofs are widely known and do not warrant repetition here.

Lemma 1 *The set of (unnormalised) states in* $\mathbb{C}^{d_A} \otimes \mathbb{C}^{d_B}$ *with Schmidt rank* r *is isomorphic to the set of* $d_A \times d_B$ *complex matrices with rank* r.

Proof Obvious from the standard proof of the Schmidt decomposition via the singular value decomposition. □

Lemma 2 *A matrix* M *has* $\operatorname{rank} M < r$ *iff all its order–*r *minors (the determinants of* $r \times r$ *submatrices) are zero.*

Proof See [17, p.13]. □

This means that a geometric characterisation of subspaces of Schmidt rank $\geq r$ is to say that the linear space $M(S)$ of associated matrices doesn't intersect the set of common zeros of all order–r minors (except in the zero vector). Such common zeroes of sets of multivariate polynomials are called (algebraic) varieties, and the one in question has been studied in the mathematical literature [24].

Definition 3 (Determinantal variety) *The affine determinantal variety* $\mathcal{D}_r(d_A, d_B)$ *over the (algebraically closed) field* $\mathbb{F}$ *in the space* $\mathbb{F}^{d_A d_B}$ *is the variety defined by the vanishing of all order–*r *minors of a* $d_A \times d_B$ *matrix, whose elements are considered as independent variables in* $\mathbb{F}$.

(Of course, in quantum theory we are mostly interested in the case $\mathbb{F} = \mathbb{C}$.)

The basic idea is now essentially parameter counting: if the dimension of S plus that of the variety $\mathcal{D}_r(d_A, d_B)$ is larger than $d_A d_B$, then the polynomial equations defining the order–r minors have roots in $M(S)$. To make this heuristic rigorous, we need to go to the corresponding projective spaces: since the polynomials defined by the minors of a matrix are homogeneous, a determinantal variety can also be thought of as a projective variety $\mathbb{P}(\mathcal{D}_r(d_A, d_B))$ in the space $\mathbb{P}^{d_A d_B - 1}$. The same is true for the subspace S, so it also has a projectification $\mathbb{P}(S)$.

Lemma 4 (Dimension of determinantal varieties) *The dimension of an affine determinantal variety is given by* $\dim \mathcal{D}_r(d_A, d_B) = d_A d_B - (d_A - r + 1)(d_B - r + 1)$.

Proof See e.g. [14, Proposition 12.2, p. 151]. □

The corresponding projective determinantal variety has, of course, dimension one less: $\dim \mathbb{P}(\mathcal{D}_r(d_A, d_B)) = d_A d_B - (d_A - r + 1)(d_B - r + 1) - 1$. Likewise, the dimension of $\mathbb{P}(S)$ is $\dim S - 1$.

Now, for projective varieties, the parameter counting argument always holds:

Lemma 5 (Intersection of projective varieties) *If* V *and* W *are projective varieties in* $\mathbb{P}^d$ *such that* $\dim V + \dim W \geq d$, *then* $V \cap W \neq \emptyset$.

Proof See e.g. [15, Theorem 6, p. 76] or [14, Exercise 11.38, p. 148]. □

Proposition 6 *For any subspace* $S \subseteq \mathbb{C}^{d_A} \otimes \mathbb{C}^{d_B}$ *of dimension* $\dim S > (d_A - r + 1)(d_B - r + 1)$, *there exists at least one state in the subspace with Schmidt rank strictly less than* r.

Proof The set of all (unnormalised) states in the bipartite space $\mathbb{C}^{d_A} \otimes \mathbb{C}^{d_B}$ forms a projective space $\mathbb{P}^{d_A d_B - 1}$ over the complex field. From Lemmas 1 and 2 and definition 3, the subset of those states with Schmidt rank less than r then forms a projective determinantal variety $\mathbb{P}(\mathcal{D}_r(d_A, d_B))$ in that space. The subspace S corresponds to the projective variety $\mathbb{P}(S)$ (a projective linear subspace), which has dimension $\dim \mathbb{P}(S) > (d_A - r + 1)(d_B - r + 1) - 1$ by assumption.

Making use of Lemma 4, we have

$$\dim \mathbb{P}(\mathcal{D}_r(d_A, d_B)) + \dim \mathbb{P}(S) \geq d_A d_B - 1 \\ = \dim \mathbb{P}^{d_A d_B - 1}.$$

Thus by Lemma 5, $\mathbb{P}(S)$ and $\mathbb{P}(\mathcal{D}_r(d_A, d_B))$ have a non-empty intersection, i.e. the subspace S contains at least one state with Schmidt rank less than r. □

Construction of highly entangled subspaces. We will now give an explicit construction of a subspace with bounded Schmidt rank that saturates the bound of Proposition 6, based on totally non-singular matrices. (Note that we can not simply take the complement of $\mathbb{P}(\mathcal{D}_r(d_A, d_B))$ in $\mathbb{P}^{d_A d_B - 1}$, since it is by no means clear that this is a projective linear variety, i.e. a subspace.)

Definition 7 (Totally non-singular matrix) *A matrix is said to be totally non-singular if all of its minors are non-zero.*

Lemma 8 *There exist totally non-singular matrices of any dimension.*

Proof The $n \times n$ Vandermonde matrix generated by $0 < \lambda_1 < \lambda_2 < \cdots < \lambda_n$ is totally positive (i.e. all its minors are strictly positive, see [18]), therefore is also totally non-singular. Alternatively, it is also clear that a generic complex matrix will be totally non-singular, as the vanishing of a minor defines a set of matrices of measure 0. □

Lemma 9 *Let M be an $m \times m$ totally non-singular matrix, with $m \geq n$. Let v be any linear combination of n of the columns of M. Then v contains at most $n-1$ zero elements.*

Proof Assume for contradiction that there exists a linear combination of n columns of M containing n or more zero elements. Let R be the set of indices of n of those zero elements and C be the set of indices of the n columns. Since there is a linear combination of the columns of M such that the elements indexed by R are all zero, the columns of the submatrix $M_{\{R,C\}}$ are linearly dependent, thus the minor $\det M_{\{R,C\}}$ is zero and we have a contradiction.□

The construction of the subspace is based on the sets of vectors introduced in Lemma 9.

Proposition 10 *Every bipartite system $\mathbb{C}^{d_A} \otimes \mathbb{C}^{d_B}$ has a subspace S of Schmidt rank $\geq r$, and of dimension $\dim S = (d_A - r + 1)(d_B - r + 1)$.*

Proof Since the bipartite states with Schmidt rank bounded by r are isomorphic to $d_A \times d_B$ matrices whose rank is at least r (Lemma 1), and a matrix has rank greater than or equal to r iff at least one of its order–r minors is non-zero (Lemma 2), it is sufficient to construct a set of linearly independent matrices S of cardinality $|S| = (d_A - r + 1)(d_B - r + 1)$ such that any linear combination of them has *at least one* non-zero order–r minor, since these then define a basis for a subspace with the desired properties.

Label the diagonals of a $d_A \times d_B$ matrix by integers k, with k increasing from lower-left to upper-right, and denote the length of the k^{th} diagonal by $|k|$. From Lemma 9, there exist sets of $t = |k| - r + 1$ linearly independent vectors of length $|k|$ such that any linear combination of them has at most $t-1$ zero elements, or conversely, has at least $|k| - (t-1) = r$ non-zero elements.

For each diagonal with length $|k| \geq r$, construct a set of linearly independent matrices S_k of cardinality $|S_k| = |k| - r + 1$ by putting these vectors down the k^{th} diagonal. By construction, any linear combination of these will have at least r non-zero elements down that diagonal. Since the determinant of the $r \times r$ submatrix with these r non-zero elements down its main diagonal is clearly non-zero, any linear combination of matrices in S_k has at least one non-zero order–r minor, thus has rank at least r.

Now define the set $S = \bigcup_k S_k$. Since matrices from different S_k have elements down different diagonals, the matrices in S are linearly independent. It remains to show that any linear combination of matrices from *different* S_k still has rank at least r. Let M be a matrix given by some linear combination of matrices in S, and let κ be the maximum k for which the linear combination includes matrices from S_k. It is still true that the κ^{th} diagonal of M must contain at least r non-zero elements. As κ labels the top-rightmost diagonal of M that contains any non-zero elements, the $r \times r$ submatrix of M with those r non-zero elements down its main diagonal is lower-triangular, so has non-zero determinant. Thus M has at least one non-zero order–r minor, so has rank at least r.

Assume for convenience that $d_B \geq d_A$. To determine the cardinality of S, i.e. the dimension of the subspace, note that a $d_A \times d_B$ matrix has $1 + d_B - d_A$ diagonals of length d_A, and 2 diagonals of each length less than d_A. Then the cardinality of S is given by

$$\begin{aligned} |S| &= \sum_k |S_k| = \sum_k (|k| - r + 1) \\ &= \big(1 + d_B - d_A\big)\big(d_A - r + 1\big) + 2 \sum_{i=r}^{d_A - 1} (i - r + 1) \\ &= (d_A - r + 1)(d_B - r + 1), \end{aligned}$$

which matches the claimed dimension of the subspace.□

Subspaces with bounded Schmidt rank. Putting together Propositions 6 and 10, we obtain the main result:

Theorem 11 *The maximum dimension of a subspace $S \subseteq \mathbb{C}^{d_A} \otimes \mathbb{C}^{d_B}$ of Schmidt rank $\geq r$ is given by $(d_A - r + 1)(d_B - r + 1)$.* □

As far as we are aware, the explicit construction of Proposition 10 is new. Refs. [10, 11] show that *generic* subspaces will also match this bound.

One could instead ask for the converse: subspaces of Schmidt rank $\leq r$. Note that geometrically this corresponds to a linear subspace lying *within* the determinantal variety $\mathcal{D}_{r+1}(d_A, d_B)$. There is a simple construction, $S = R \otimes \mathbb{C}^{d_B}$, for any subspace $R \subseteq \mathbb{C}^{d_A}$ of dimension r, which achieves $\dim S = r d_B$. This is clearly tight if $r = 1$ or $r = d_A$. In fact, one can show that this construction is optimal in general, which is immediate from the following theorem due to Flanders [20]:

Theorem 12 (Flanders) *Let S be a subspace of the space of $d_A \times d_B$ matrices, where $d_A \leq d_B$. Let r be the maximum rank of any element of S. Then $\dim S \leq r d_B$.* □

Another interesting variant is to ask what are the subspaces which have Schmidt rank *exactly* r. For example, our construction above yields subspaces of dimension $d_B - d_A + 1$ in $\mathbb{C}^{d_A} \otimes \mathbb{C}^{d_B}$ of Schmidt rank equal to d_A. A different example is given by the three-dimensional completely antisymmetric subspace of $\mathbb{C}^3 \otimes \mathbb{C}^3$, which has Schmidt rank equal to 2. This question has been the subject of a remarkably long-running study in the linear algebra literature and, as far as we are aware, the general case remains unsolved. The best existing results are summarised in the following theorem, which can be found in [21]:

Theorem 13 (Westwick) *Let S be the largest subspace of the space of $d_A \times d_B$ matrices, with $d_B \geq d_A$, such that the rank of every non-zero element of S is r. Then in general,*

$$d_B - r + 1 \leq \dim S \leq d_A + d_B - 2r + 1.$$

Furthermore, if $d_B - r + 1$ does not divide $(d_A - 1)!/(r-1)!$, then $\dim S = d_B - r + 1$. If $d_A = r + 1, d_B = 2r - 1$, then $\dim S = r + 1$. □

For sufficiently large d_B, it is of course impossible for $d_B - r + 1$ to divide $(d_A - 1)!/(r - 1)!$, so the result for that case applies to all sufficiently high-dimensional spaces. It appears to be impossible to obtain this result via general dimensional arguments similar to those used in this paper, which only reproduce the general upper bound, $\dim S \leq (d_B - r + 1) + (d_A - r)$.

Discussion: applications and open questions. We have determined the exact maximum dimension of subspaces of Schmidt rank $\geq r$ in any bipartite quantum system. The upper bound on the dimension is a generalisation of Proposition 1.4 of Ref. [8] for a subspace avoiding the manifold of product states, to the avoiding of a determinantal variety. Our constructive lower bound seems to differ from Parthasarathy's (in the case $r = 2$), which is based on unextendible product vector systems.

Comparing these results, using the Schmidt measure, with [22], where the entropy measure of entanglement is used, we have much tighter control on the entanglement in subspaces. For example, in the cited paper, the random subspaces that are constructed are necessarily highly entangled, simply because that is the generic behaviour of random states. In contrast, here we find the largest subspaces of bounded Schmidt rank over the *whole range* of the entanglement measure, including values far away from typical. This is most clearly demonstrated by considering subspaces with Schmidt rank within a constant fraction of the maximum: $r \geq k d_A$. For $k \geq 2^{-d_A/(d_B \ln 2)}$, using the results of [22, Theorem IV.1] gives nothing better than the trivial one-dimensional subspace, yet the exact result is asymptotically of order $(1-k)^2 d_A d_B$, i.e. within a constant fraction of the entire space!

Our results can be used, in the spirit of [22], to construct highly mixed states of very large Schmidt measure [7]: let ρ be the normalised projector onto a maximum dimensional subspace S of Schmidt rank $\geq r$. Then, since every pure state decomposition of ρ can only consist of state vectors from S, any entanglement measure built from the Schmidt ranks of the constituent pure states has to be at least r. For example, in arbitrarily large $d \times d$–systems, we thus find for any p states of rank $\geq p^2 d^2$ (i.e. entropy $2 \log d + 2 \log p$) and Schmidt measure $\geq (1-p)d$. The ideas and results described in this paper also have applications to the infamous additivity problem in quantum information theory. Specifically, they can be used to construct a counter-example to additivity of the minimum output Renyi 0-entropy of a quantum channel [12].

The bipartite results naturally beg the question: what about multipartite generalisations? For example, we might consider subspaces with constraints on the Schmidt rank across some or all bipartite cuts. In fact, this case is already answered by the bipartite results. Clearly, the subspace dimension is upper-bounded by Proposition 6 applied to whichever constraint gives the tightest restriction on the dimension. However, since generic subspaces saturate Proposition 6, a generic subspace with this dimension will also satisfy all the other less restrictive constraints. Therefore, the trivial multipartite bound, given directly by the bipartite results, is already tight.

Note that the case of multipartite completely entangled subspaces was discussed in Refs. [8, 9], and the dimension formula given there would seem to contradict the above discussion. But they define a completely entangled subspace in the multipartite setting to be one which contains no *fully* separable states, i.e. every state in the subspace must be entangled across *at least one* bipartite cut. This is of course different to requiring that states in the subspace obey a set of rank constraints across bipartite cuts *simultaneously*, so there is in fact no contradiction.

We could also attempt to make statements about more operationally motivated entanglement measures, especially those based on von Neumann or Rényi entropies, as for example in Ref. [22]. However, the algebraic techniques used here do not seem to give any insight into these problems.

Acknowledgments. The authors acknowledge support by the European Commission, project "QAP", and the U.K. EPSRC through postgraduate scholarships, the "QIP IRC" and an Advanced Research Fellowship.

We thank Aram Harrow, Richard Jozsa, Richard Low and Will Matthews for various spirited discussions about the content of this paper; especially the latter three for their solution of the $r = d_A = d_B$ case, which was the starting point of this work. We would also like to thank Wee Kang Chua for posing the question of subspaces with *upper*-bounded Schmidt rank.

Note added. Since submitting this paper, Walgate and Scott [19] have given a more explicit proof than that contained in Refs. [10, 11] showing that generic subspaces saturate the dimension bound for the $r = 2$ case of completely entangled subspaces.

[1] C. H. Bennett, G. Brassard, et al., Phys. Rev. Lett. **70**, 1895 (1993).

[2] C. H. Bennett and G. Brassard, in *Proc. IEEE Int. Conf. Computers, Systems and Signal Processing, Bangalore, India* (1984), pp. 175–179.
[3] A. K. Ekert, Phys. Rev. Lett. **67**, 661 (1991).
[4] P. W. Shor, in *Proc. 35th Annual Symp. Foundations of Computer Science*, edited by S. Goldwasser (Los Alamitos, CA, 1994), pp. 124–134.
[5] R. Horodecki, P. Horodecki, et al., `quant-ph/0702225` (2007).
[6] M. Christandl, Ph.D. thesis, Cambridge (2006).
[7] B. M. Terhal and P. Horodecki, `quant-ph/9911117`; Phys. Rev. A **61**, 040301(R) (2000).
[8] K. R. Parthasarathy, `quant-ph/0307182`; Proc. Indian Acad. Sci. (Math. Sci.) **114**, 365 (2004).
[9] N. R. Wallach, Contemporary Math. **305**, 291 (2002).
[10] D. Eisenbud, Am. Journ. Math. **110**, 541 (1988).
[11] B. Ilic and J. M. Landsberg, Math. Ann. **314**, 159 (1999).
[12] T. S. Cubitt, A. Harrow, D. Leung, A. Montanaro, and A. Winter, in preparation (2007).
[13] P. Griffiths and J. Harris, *Principles of Algebraic Geometry* (John Wiley and Sons, New York, 1978).
[14] J. Harris, *Algebraic Geometry: A First Course* (Springer-Verlag, 1995).
[15] I. Shafarevich, *Basic Algebraic Geometry 1: Varieties in Projective Space* (Springer-Verlag, 1994), 2nd ed.
[16] D. Cox, J. Little, and D. O'Shea, *Ideals, Varieties, and Algorithms* (Springer-Verlag, 2007), 3rd ed.
[17] R. A. Horn and C. R. Johnson, *Matrix Analysis* (Cambridge University Press, 1985).
[18] S. M. Fallat, Amer. Math. Monthly **108**, 697 (2001).
[19] J. Walgate and A. J. Scott, arXiv:0709.4238 (2007).
[20] H. Flanders, J. London Math. Soc. **37**, 10 (1962).
[21] R. Westwick, Linear and Multilinear Algebra **20**, 171 (1987).
[22] P. Hayden, D. Leung, and A. Winter, `quant-ph/0407049`; Comm. Math. Phys. **265**, 95 (2006).
[23] H. Chen, J. Math. Phys. **47**, 052101 (2006).
[24] Determinantal varieties have also been studied in quantum information theory by Chen [23], in a different context.